Nonparametric Methods for Repair Models: Frequentist and Bayesian

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Abstract

Statistical analyses of repair models are often concerned with the distribution of the time to first repair. From that distribution and knowledge of the repair scheme, one can derive other results such as the expected number of repairs by time t, and so forth. Various nonparametric procedures have been developed for inference about F. We discuss various frequentist and Bayesian approaches, and present some recent results. The results apply to a number of models including those of Brown and Proschan (1983) and Block, Borges and Savits (1985).

1 Introduction

Consider a system consisting of one unit. Let the lifetime distribution of the unit be F. When the unit fails in the field, say, at time X_1, cost considerations typically preclude replacement with a brand new unit, though this would be preferable to maintain high reliability. Instead, the unit is repaired and put back in use. A *repair model* postulates the conditional distribution of the life, X_2, of the repaired unit can depend on previous failure times of the unit, any covariates observed till the last failure time and on F. The repair process can be repeated and we observe $S_1, S_2, \dots$ the failure times of the system. Note that $X_1 = S_1, X_2 = S_2 - S_1, \dots$ are the inter-failure times of the system which are also the same as the extra amounts of life given to the system after each repair. Let $Y_1, Y_2, \dots$ be another sequence of random variables which evolve with the system and are observed after each repair. These may be viewed as environmental variables, "strengths" or "degrees" of repairs, etc. Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n, Y_1, \dots, Y_n)$ be the σ-field of the random variables available till the time of the nth failure. It is of interest to estimate the life distribution F (and associated cumulative hazard function and failure rate) of the time to first failure of a brand new unit using data on repeated failures of such a system subject to a model of repair.

2 Repair models

For $a \geq 0$, the residual life distribution $F_a(x)$ is defined by

$$\bar{F}_a(x) = 1 - F_a(x) = \frac{\bar{F}(a+x)}{\bar{F}(x)}, \text{ for } x > 0. \tag{1}$$

A general repair model may be specified as follows, using the concept of effective ages:

$$P(X_1 \leq x) = F(x),$$

and

$$P(X_n > x | \mathcal{F}_{n-1}) = \bar{F}_{A_n}(x), n = 2, \dots$$

where A_n, called the effective age after the $(n-1)$st failure, is $\mathcal{F}_{n-1}$-measurable, $n = 2, 3, \dots$. Note that $\mathcal{F}_0$ is the trivial σ-field and we set $A_1 = 0$.

We state below some of the standard repair models in terms of the definition above. In each case, one just has to specify the sequence of effective ages $\{A_n\}$.

A "*Replace with a new unit on failure*" model corresponds to setting the effective ages A_n to be 0 for all n.

A "*Perform minimal repair forever*" model, corresponds to putting $A_n = X_{n-1}, n = 2, 3, \ldots$.

For the Brown and Proschan (1983) randomized minimum repair model (*BP model*), one chooses a number p in $(0, 1)$ and auxiliary random variables $Y_1, Y_2, \ldots$ which are i.i.d. uniform and defines

$$A_n = \begin{cases} 0 & \text{if } Y_{n-1} \leq p \\ X_{n-1} & \text{if } Y_{n-1} > p \end{cases},$$

for $n = 2, 3, \ldots$

For the Block-Borges-Savits (1985) randomized minimal repair model (*BBS model*), which extends the Brown-Proschan model, one employs a function $p(\cdot)$ on $(0, \infty)$ taking values in [0,1] and defines

$$A_n = \begin{cases} 0 & \text{if } Y_{n-1} \leq p(X_{n-1}) \\ X_{n-1} & \text{if } Y_{n-1} > p(X_{n-1}) \end{cases}.$$

Kijima (1989) introduced models that feature "degree of repair" random variables which allow for repairs that can be described as being somewhere between perfect and minimal repairs, that is, repairs that could do better than minimal repair without necessarily restoring the unit to a state equivalent to that of a new unit. In Kijima's (1989) models I and II, the effective age A_n, $n = 1, 2, \ldots$, to which the system is restored after repair depends not only in its age just before failure but also on the degree of repair random variables D_n, $n = 1, 2, \ldots$. It will be assumed that the D's are independently distributed on $[0, 1]$ and independent of other processes.

The effective age A_n at the time of the $(n-1)^{\text{th}}$ failure will depend only on $\mathcal{F}_{n-1} \stackrel{def}{=} \sigma(X_1, \ldots, X_{n-1}, D_1, \ldots, D_{n-1})$. We will define $A_1 = 0$. The distribution of X_n, the n^{th} interfailure time given $\mathcal{F}_{n-1}$, will depend only on the effective age A_n.

Kijima's model I defines A_n by

$$A_n = \sum_{i=1}^{n-1} D_i X_i, \ n > 1.$$

Note that with this specification

$$A_{n+1} = A_n + D_n X_n$$

and the A_n's are increasing and $A_{n+1} \leq A_n + X_n$ indicating that a better than minimal repair has been performed.

Kijima's model II defines A_n by

$$A_n = \sum_{k=1}^{n-1} \left(\prod_{i=k}^{n-1} D_i \right) X_k, \ n > 1.$$

Note that the effective ages satisfy

$$A_{n+1} = D_n(A_n + X_n) \tag{2}$$

which is less than $A_n + X_n$, once again indicating better than minimal repair.

When data from the BBS model are observed till the time of the $n + 1$st perfect repair, i.e. when n independent sequences of observations starting from a brand new unit each time, are obtained, Whitaker and Samaniego (1989) and Hollander, Presnell and Sethuraman (1992) proposed frequentist estimates for F. They also gave the asymptotic distributions, tests for $F = F_0$, asymptotic confidence intervals for F, etc.

For the above frequentist methods of estimation of F the data acquisition is method is structured to obtain n i.i.d. sequences of observations. This is an unnatural requirement. In the next section, we propose a Bayesian method to estimate F for the BBS model where observation does not have to end before the next perfect repair.

3 Bayesian Methods in Repair Models

We will use the symbol P to denote the probability measure associated with F and the symbol Pr to denote the joint probability distribution of the the variables under consideration. The Bayesian method to estimate F or P from data based on a general repair model will require the use of a prior distribution for the distribution function F. It will be convenient to restate this as the Bayesian estimation of a probability measure P on a general state space $(\mathcal{X}, \mathcal{A})$ where the data $X_1, \ldots, X_n$ arise from a repair model, properly defined at this level of generality, to include the repair models mentioned in the previous section. The class of probability measures on $(\mathcal{X}, \mathcal{A})$ will be denoted by $\mathcal{P}$. The Bayesian methods will consist of three steps:

[**1**] Definition of suitable classes of prior distributions for P.

[**2**] Definition of general repair models.

[**3**] Calculation of posterior distribution of P given the data.

For any $P \in \mathcal{P}$ and non-empty set $B \in \mathcal{A}$, define the restricted probability measure P_B as

$$P_B(A) = \begin{cases} \frac{P(B \cap A)}{P(B)} & \text{if } P(B) \neq 0 \\ \phi_B(A) & \text{if } P(B) = 0 \end{cases}$$

for all $A \in \mathcal{A}$.

A general repair model may be specified as follows, using the concept of *effective restriction sets* $A_n, n = 1, 2, \ldots$ as follows:

$$Pr(X_1 \in A) = P(A),$$

and

$$Pr(X_n \in A | \mathcal{F}_{n-1}) = P_{A_n}(A)$$

$n = 2, \ldots$, where A_n is a measurable set in $\mathcal{F}_{n-1}$. The choice of the random effective restriction sets A_n, depending on the previous data, defines the repair model. This includes all the models described in Section 2. This completes the description of step **2** in the Bayesian approach.

One can think of arbitrary priors for P to complete step **1**, but this will not allow us to complete the main step, namely step **3**. Thus we should choose priors α for P so that, if

$$P \sim \alpha, \text{ and } A \text{ is a measurable set and } X|P \sim P_A,$$

then we should be able to calculate the distribution of P given X, i.e.

$$Pr(P \in B|X).$$

If A were the whole space $\mathcal{X}$, then this calculation is easy if we use Dirichlet priors. This problem is already difficult if A is a proper subset, which is the case in repair models. We solve this problem and extend it by showing that we can use a larger class of priors, called partition based priors which include Dirichlet and generalized Dirichlet priors, and still carry out the calculation of posteriors under general repair models. Two examples of comparisons of the Bayesian estimates obtained this way with the frequentist Whitaker-Samaniego estimate in the BBS repair model are given in the graphs below. The first example uses 13 observations from a χ^2 distribution with 5 degrees of freedom and with 3 perfect repairs. The second example is based on the reduced Proschan airconditioner data; it consists of 24 observations and 4 perfect repairs. In both cases the prior distribution of P was assumed to be a Dirichlet with parameter $\gamma\beta(\cdot)$ where β is an exponential distribution with a parameter equal to the reciprocal of the sample mean, and the constant γ was taken to be 4. The figures do not change very much if we vary the constant γ around 4.

An important point to note is that while the Whitaker-Samaniego estimate requires that the data stops as soon as it is decided that the next repair will be a perfect repair, the Bayesian method has no such artificial restriction on the data.

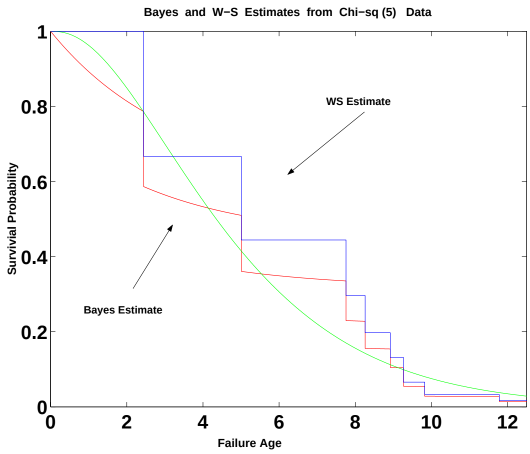

Figure 1: Comparison of the Bayes and W-S estimates from observations from a χ^2 distribution

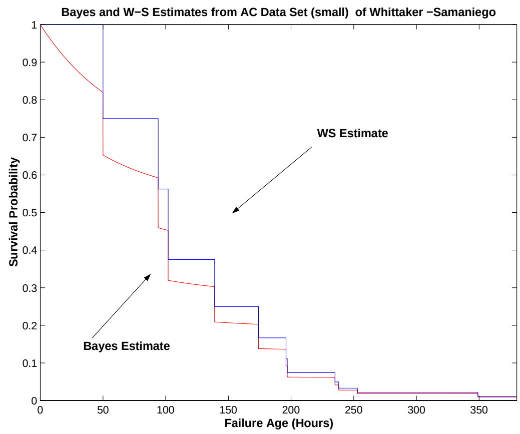

Figure 2: Comparison of the Bayes and W-S estimates from the reduced Proschan air conditioner data

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